

# Classification and detection of symmetry fractionalization in chiral spin liquids

Lukasz Cincio<sup>1</sup> and Yang Qi<sup>2,1</sup>

<sup>1</sup>*Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada*

<sup>2</sup>*Institute for Advanced Study, Tsinghua University, Beijing 100084, China*

In this work we study the crystal symmetry fractionalization in chiral spin liquids with the chiral-semion topological order. We show that if such a chiral spin liquid is realized in a two-dimensional lattice model with odd number of spin- $\frac{1}{2}$  per unit cell and the state preserves spin rotation symmetry and translation symmetries, the semion excitation must carry both half-integer spin and fractional crystal quantum numbers. As a result, only a unique symmetry enriched topological phase can be realized in chiral spin liquids in a spin- $\frac{1}{2}$  kagome lattice model. These fractional symmetry quantum numbers are confirmed numerically using ground state wave functions obtained with the density matrix renormalization group method.

## I. INTRODUCTION

Quantum spin liquids<sup>1,2</sup> with topological orders<sup>3-6</sup> have generated considerable interests in the study of strongly correlated quantum systems, because they represent a class of quantum phases beyond the Landau's paradigm of symmetry breaking phases<sup>7</sup>, and they are closely related to the study of high- $T_c$  superconductivity<sup>8,9</sup>. So far many different types of quantum spin liquids, including gapped and gapless spin liquids, have been studied. Among these different types, the chiral spin liquid state<sup>10,11</sup> was the earliest one to be proposed theoretically. The chiral spin liquid state has been constructed in exactly solvable model Hamiltonians<sup>12</sup>, and it was recently found in a variety of spin- $\frac{1}{2}$  models with short-range interaction on the kagome lattice by different numerical and theoretical approaches, including the density matrix renormalization group (DMRG) method<sup>13-17</sup>, fermionic parton constructions and variational Monte Carlo method<sup>18,19</sup> and theoretical constructions<sup>20,21</sup>.

One question remains to be answered is whether the chiral spin liquid states obtained in different methods are the same, i.e. whether they belong to the same phase. This question can be easily answered for topologically trivial phases because these phases are classified by the symmetry the ground states have, and can be detected by measuring correlation functions that detects all possible ways of symmetry breaking<sup>7</sup>. However, classifying and detecting topological orders are much harder because these orders are not reflected in any correlation functions of local observables.

First, the chiral spin liquid states have an intrinsic topological order, which is characterized by topological excitations carrying fractional statistics, and chiral central charges of the gapless edge states. In particular, the chiral spin liquids have the “chiral-semion” topological order, which is the same as the intrinsic topological order of a  $\nu = \frac{1}{2}$  bosonic fractional quantum hall (FQH) state<sup>10,22</sup>. (In this work, the terminology of “chiral spin liquid” only refers to spin liquids with the chiral-semion topological order. In some literatures<sup>23,24</sup>, chiral spin liquids also refer to time-reversal-symmetry-breaking spin liquids with  $\mathbb{Z}_2$  topological order, and these are differ-

ent from the chiral spin liquids discussed in this work.) In this topological order there are two types of anyons: the trivial anyon (denoted by  $\mathbb{1}$ ) and the semion  $s$ . The semion  $s$  has a fractional statistics such that exchanging two semions gives a statistical phase of  $e^{i\pi/2} = i$ , and it satisfies the fusion rule that two semions fuse into a trivial anyon  $s \times s = \mathbb{1}$ . The intrinsic topological order can be detected numerically: the number of anyons can be measured from the topological ground state degeneracy on a torus<sup>25,26</sup> and topological entanglement entropy<sup>27,28</sup>, while the anyon statistics can be measured from the modular matrices<sup>29-31</sup>. These methods have been applied to identify the chiral-semion topological order in the DMRG studies<sup>14-17,30</sup>.

Second, the crystal and spin rotation symmetries that are present in the chiral spin liquid states further enriches their topological order<sup>32</sup>. Because there is only one type of nontrivial anyon, the symmetry enriched topological (SET) order is classified by the symmetry fractionalization carried by the semion excitations<sup>33-36</sup>. Here, the symmetry fractionalization refers to the phenomenon that an anyon excitation can carry a projective representation of the symmetry group and consequently fractional symmetry quantum numbers<sup>35-42</sup>. This occurs in SET phases because, although any physical state must carry a linear representation of the symmetry group, it also contains a collection of anyons that fuse into  $\mathbb{1}$ . Therefore, each anyon can carry a projective representation as long as the representations carried by the collection of anyons always fuse into a linear representation. For example, in a chiral spin liquid, the semions must appear in pairs in a physical state because of the fusion rule  $s \times s = \mathbb{1}$ . Therefore, although each physical state must carry a linear representation of the  $\text{SO}(3)$  spin rotation symmetry group, the semion can carry a projective representation, and therefore carries a half-integer spin.

The symmetry fractionalization can be classified by the projective representation of the symmetry group according to the anyon fusion rule<sup>35-37</sup>. However, not all the symmetry fractionalization obtained this way can be realized in a two-dimensional (2D) system, as many of them are anomalous and can only be realized on a surface of a three-dimensional (3D) symmetry-protected topologi-

cal (SPT) phase<sup>43</sup>. For on-site unitary symmetries, these anomalous symmetry fractionalizations can be systematically identified<sup>37,44</sup>. However, for antiunitary and crystal symmetries such general methods have not yet been established and anomalous symmetry fractionalization has only been studied case-by-case<sup>45–48</sup>.

Since the information of symmetry fractionalization can distinguish different symmetric spin liquid phases, the detection of symmetry fractionalization is essential to the study of spin liquid states. Methods of numerically and experimentally detecting the symmetry fractionalization have been developed in the context of  $\mathbb{Z}_2$  spin liquids<sup>33,49–53</sup>, and they can be generalized to be applied to the chiral spin liquids.

In this work we answer the aforementioned question by studying the crystal symmetry fractionalization in 2D chiral spin liquids with the chiral-semion topological order. Particularly we find there is only one way to fractionalize the crystal symmetries, and all the chiral spin liquids found in previous studies must belong to the same phase. We first discuss in general how to define fractionalized crystal symmetry quantum numbers for anyons with fractional statistics in Sec. II. We then classify crystal symmetry fractionalization in 2D chiral spin liquids in Sec. III. In this work, we consider spin-rotational-invariant chiral spin liquids with odd number of spin- $\frac{1}{2}$  per unit cell (this includes all the chiral spin liquids found in different lattice models on the kagome lattice). Under this assumption, there is only one nonanomalous combination of fractional symmetry quantum number the semion can take. Therefore, there is only one type of symmetric chiral spin liquid that can be realized in a spin- $\frac{1}{2}$  model on the kagome lattice. In Sec. IV, we numerically study the symmetry fractionalization in ground state wave functions obtained using the DMRG method, and confirm that the chiral spin liquid state indeed has the symmetry fractionalization determined in Sec. III. We conclude in Sec. V.

## II. CRYSTAL SYMMETRY FRACTIONALIZATION OF ANYONS

In this section we discuss the definition of crystal symmetry fractionalization for anyons carrying fractional statistics, and pave the way for studying the classification of chiral spin liquids with crystal symmetries in the rest of the paper. Here, we restrict ourselves to Abelian topological orders.

The concept of crystal symmetry fractionalization was first studied<sup>33</sup> in the context of analyzing the projective symmetry group (PSG) of mean field ansatz in parton constructions, where the partons carry a projective representation of the crystal symmetry group, and thus fractional crystal symmetry quantum numbers. Although the PSG analysis is useful in classifying symmetric spin liquids in parton constructions (including gapless spin liquids that are beyond the scope of gapped SET phases),

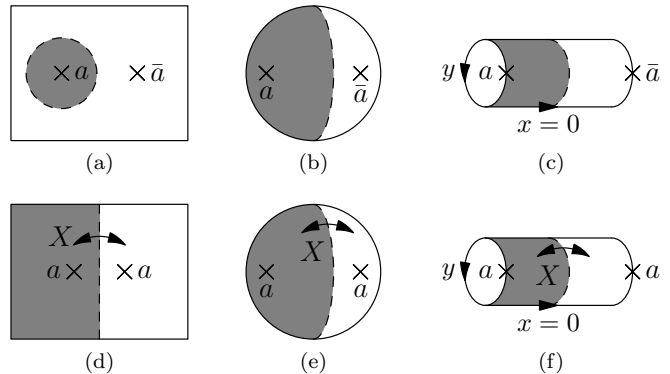


FIG. 1. (Color online) Symmetry actions on anyons. The “x” symbols mark the locations of anyons. The shaded area and the unshaded area mark the two regions in the Schmidt decomposition in Eqs. (1) and (3). In Figs. (a)-(c), an antiunitary symmetry  $X$  maps  $a$  and  $\bar{a}$  to themselves; in Figs. (d)-(f), a unitary symmetry  $X$  maps  $a$  to  $\bar{a}$  and vice versa. (a) A state that contains two anyons  $a$  and  $\bar{a}$  is shown. Each anyon is symmetric under an antiunitary symmetry  $X$ . (b) The configuration in (a) is smoothly deformed to a sphere. (c) The configuration in (b) is smoothly deformed to an infinite cylinder. (d) A configuration containing two anyons:  $a$  and  $\bar{a}$ , which are mapped into each other by a unitary symmetry  $X$ . (e) The configuration in (a) is smoothly deformed to a sphere. (f) The configuration in (b) is smoothly deformed to an infinite cylinder.

it has limitations when being used to classify symmetry fractionalization in SET phases. First, the parton used in a construction usually corresponds to only one type of anyon, so it does not explicitly give the full symmetry fractionalization if there are more than one type of anyons (this is not an issue for the chiral-semion topological order). Second, the projective symmetry representation carried by the parton may not be the same as the projective representation of the corresponding anyon, as we shall demonstrate with an explicit example in Appendix A.

For SET phases with the  $\mathbb{Z}_2$  (toric code) topological order, previous works<sup>35,52–54</sup> have provided practical definitions of crystal symmetry fractionalization and methods to measure it from the ground state wave functions. Here we briefly review these definitions and measurements, and generalize them to anyons taking fractional statistics other than the Bose and Fermi statistics, like the semions in the chiral-semion topological order.

The key difficulty in defining the projective symmetry representation carried by an anyon is that the anyon cannot appear alone in a physical wave function. Instead, a physical wave function must contain anyons which can be fused together as a trivial anyon. For example, in the chiral-semion phase the semions must appear in pairs as  $s \times s = 1$ . Here we take the assumption of symmetry localization<sup>35,37</sup>, which states that the symmetry action on a physical wave function, which is always well defined, can be fractionalized into actions localized in the neigh-

borhoods of the anyons. To define the symmetry action on an individual anyon, we take a wave function  $|\Psi\rangle$  containing two well-separated anyons  $a$  and  $\bar{a}$  that fuses into the trivial particle  $a \times \bar{a} = \mathbb{1}$ . Then, we construct a Schmidt decomposition between the region containing  $a$  and the region containing  $\bar{a}$  [see Fig. 1(a)],

$$|\Psi\rangle = \sum_{\alpha} \lambda_{\alpha} |a(\mathbf{r})\rangle_{\alpha} \otimes |\bar{a}(\infty)\rangle_{\alpha}. \quad (1)$$

In this Schmidt decomposition,  $|a(\mathbf{r})\rangle_{\alpha}$  are the Schmidt eigenstates supported by the region containing an anyon  $a$  at  $\mathbf{r}$ , and  $|\bar{a}(\infty)\rangle_{\alpha}$  are the Schmidt eigenstates supported by the region containing an anyon  $\bar{a}$  at infinity. The symmetry action on the anyon can then be described by how the Schmidt eigenstates transform under a symmetry operation  $X$ ,

$$X : |a(\mathbf{r})\rangle_{\alpha} \rightarrow U_{\alpha\beta}^X |a(\mathbf{r})\rangle_{\beta}, \quad (2)$$

where  $U^X$  is the symmetry representation carried by the Schmidt eigenstates  $|a(\mathbf{r})\rangle_{\alpha}$ . It is a projective representation with  $U(1)$  coefficients because of the  $U(1)$  gauge degree of freedom present in the Schmidt decomposition in Eq. (1). Here we assume that a) a symmetry breaking field is applied to the neighborhood of  $a$  and  $\bar{a}$  to lift any symmetry-protected degeneracy of the anyons, and b) an infinitesimal symmetry breaking field is applied near the boundary of the decomposition to lift any symmetry-protected degeneracy in the entanglement spectrum. Since  $a$  is an Abelian anyon, the Schmidt decomposition in Eq. (3) does not have any degeneracy after all symmetries are broken. Therefore, the gauge degree of freedom in the Schmidt decomposition is  $U(1)$ .

This projective representation can be directly used to define the quantum number fractionalization<sup>52</sup>  $X^2 = \pm 1$  for an antiunitary  $\mathbb{Z}_2$  symmetry operation  $X$  (like time-reversal). This is because the arbitrary  $U(1)$  phase factor in the projective representation is canceled when  $X$  is applied twice. In other words, for an antiunitary group  $\mathbb{Z}_2^X$ ,  $H^2(\mathbb{Z}_2^X, U(1)) = H^2(\mathbb{Z}_2^X, \mathbb{Z}_2)$ . Here,  $H^2(G, R)$  denotes the second group cohomology with coefficient  $R$ <sup>43</sup>. The commutation relation fractionalization<sup>52</sup> between two unitary symmetry operators can also be defined in this way.

However, this projective representation cannot be used directly to define the quantum number fractionalization  $X^2 = \pm 1$  for a unitary  $\mathbb{Z}_2$  symmetry  $X$  like inversion and mirror reflection, because, due to the arbitrary  $U(1)$  phase factor, the value of  $X^2$  can smoothly interpolate between  $+1$  and  $-1$ . In other words,  $H^2(\mathbb{Z}_2^X, U(1))$  is always trivial for a unitary group.

Instead, this symmetry fractionalization can be defined using the symmetry eigenvalue of a wave function containing two anyons located at two symmetry-related positions<sup>52,54</sup>, as shown in Fig. 1(d). For anyons in the  $\mathbb{Z}_2$  topological order, the symmetry fractionalization of a bosonic anyon is defined simply using the symmetry parity eigenvalue of the two-anyon wave function: the

bosonic anyon carries trivial (nontrivial) symmetry fractionalization if the two-anyon wave function is symmetric (antisymmetric) under the symmetry, respectively. However, for fermionic anyons an opposite definition was adapted: the fermionic anyon carries trivial (nontrivial) symmetry fractionalization if the two-anyon wave function is antisymmetric (symmetric), because for a physical fermion without any symmetry fractionalization the two-fermion wave function would be antisymmetric. For an anyon with fractional statistics (like a semion) there is no natural expectation of the parity of the two-anyon wave function. For consistency, we choose to generalize the definition for bosonic anyons to general Abelian anyons. In other words, regardless of the self-statistics of the anyon, we always define the symmetry fractionalization to be trivial (nontrivial) if the two-anyon wave function is symmetric (antisymmetric) under the symmetry  $X$ .

The symmetry fractionalization defined above for both antiunitary and unitary  $\mathbb{Z}_2$  symmetries can be directly related to 1D SPT invariants if the system is put on a infinite cylinder and viewed as a quasi-1D system. This is discussed in the context of the  $\mathbb{Z}_2$  topological order by Zaletel *et al.*<sup>53</sup>, and it can be generalized to other topological orders. This relation to 1D SPT invariants will be used in Sec. III to identify anomalous combinations of fractional symmetry quantum numbers, and in Sec. IV to numerically detect the symmetry fractionalization from ground state wave functions.

On an infinite cylinder, the ground states of a topologically ordered system with  $n$  types of anyons are  $n$ -fold degenerate (the degeneracy is only approximate when the width of the cylinder is finite). In the subspace formed by the degenerate ground states, a special basis called the minimum entropy states<sup>29</sup> can be chosen. In this basis, each ground state corresponds to one type of anyon flux threading through the cylinder. It is denoted by  $|a\rangle$ , where  $a$  labels the type of anyon. Viewed as a quasi-1D system, the state  $|a\rangle$  is a trivial (nontrivial) SPT state protected by a  $\mathbb{Z}_2$  symmetry  $X$  if the anyon  $a$  carries a trivial (nontrivial) symmetry fractionalization  $X^2 = \pm 1$ , respectively. The way the crystal symmetry  $X$  is set up depends on whether  $X$  is unitary or antiunitary: an antiunitary symmetry should be implemented such that it preserves the longitudinal direction  $Xx = x$ , while a unitary symmetry should reverse the longitudinal direction  $Xx = -x$ , where  $x$  denotes the longitudinal coordinate, as shown in Fig. 1. These setups are consistent with the fact that both on-site antiunitary symmetries and unitary reflection symmetries protect nontrivial 1D SPT states, and the SPT invariant is encoded in the way the symmetry acts on the Schmidt eigenstates obtained from a Schmidt decomposition of the ground state, when the cylinder is cut into left and right parts at  $x = 0$ <sup>55</sup>,

$$|\Psi\rangle = \sum_{\alpha} \lambda_{\alpha} |L\rangle_{\alpha} \otimes |R\rangle_{\alpha}. \quad (3)$$

For an antiunitary symmetry, the Schmidt decomposi-

tion in Eq. (3) can be smoothly deformed to the one in Eq. (1), as shown in Fig. 1(c), and both the 1D SPT and symmetry fractionalization is defined using the double action of  $X$  on the left Schmidt eigenstates.

For a unitary symmetry, the symmetry operator  $X$  maps the left part to the right part and vice versa. This mapping is symmetric (antisymmetric) if the state is a trivial (nontrivial) SPT. If we consider instead a finite cylinder with two open boundaries on both sides [see Fig. 1(f)], each boundary then hosts an anyon  $a$ , and it can be shown that the  $X$ -parity of this two-anyon wave function is determined by how  $X$  maps between left and right Schmidt states. If the mapping is symmetric (antisymmetric), the wave function is even (odd) under  $X$ . Therefore, according to our definition, the anyon carries a trivial (nontrivial) symmetry fractionalization of  $X^2 = \pm 1$  if  $|a\rangle$  is a trivial (nontrivial) 1D SPT, respectively.

We note that our definition of symmetry fractionalization for a unitary  $\mathbb{Z}_2$  symmetry is consistent regardless of the anyon statistics, and it is opposite to the definition previously used for fermionic anyons<sup>35,52,54</sup>. It is also different from the projective representation of a fermionic parton operator<sup>33,52,54</sup>. Using this definition, the classification of symmetry fractionalization always directly maps to 1D SPT classification, where a trivial (nontrivial) symmetry fractionalization corresponds to a trivial (nontrivial) 1D SPT. Furthermore using this definition it is easy to derive the fractional quantum number of an anyon bound state using the fusion rule. If two anyons fuse into a third anyon  $a \times b = c$ , the fractional quantum number of  $c$ ,  $X_c^2$ , is directly obtained by multiplying the fractional quantum numbers of  $a$  and  $b$ ,  $X_a^2$  and  $X_b^2$ :

$$X_c^2 = X_a^2 X_b^2, \quad (4)$$

without the need to include a twist factor depending on the anyon statistics<sup>35,52,54</sup>. Although we will not use Eq. (4) in this work (because there is only one type of nontrivial anyon in the chiral-semion topological order), this fusion rule and our definition of anyon symmetry fractionalization will be useful in the study of crystal symmetry fractionalization in more complicated topological orders, like the double-semion topological order.

### III. CLASSIFICATION OF 2D CHIRAL SPIN LIQUID

Using the definition of crystal symmetry fractionalization described in the previous section, here we study the classification of crystal symmetry fractionalization in a 2D chiral spin liquid with the chiral-semion topological order. The key result of this section is the following: if the semion excitation carries a half-integer spin, it must also carry fractionalized crystal symmetry quantum numbers. In particular, only one type of fully symmetric chiral spin liquid can be realized on a 2D kagome lattice.

We begin with enumerating the possible ways to fractionalize crystal symmetries in a chiral spin liquid on the kagome lattice. In a chiral spin liquid state the time-reversal and mirror symmetries are broken by the spin chirality order parameter  $\mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k)$ , which is odd under both time-reversal and mirror symmetries. However the combination of mirror and time-reversal is still a symmetry operation. Therefore the symmetry group is reduced from  $G = p6m \times \mathbb{Z}_2^T$  (which is generated by two translations  $T_{1,2}$ , two mirror reflections  $\mu$  and  $\sigma$ , and time-reversal operation  $T$ , as shown in Figs. 2 and 3) to  $p6m^*$ , which is the group with the identical structure as  $p6m$  but the mirror operations  $\mu$  and  $\sigma$  are replaced by antiunitary operators  $\mu^* = \mu T$  and  $\sigma^* = \sigma T$ , respectively.

Quantum number	Algebraic relation	Anomaly-free choice
$\omega_{12}$	$T_1 T_2 = \pm T_2 T_1$	-1
$\omega_{\mu^*}$	$(\mu^*)^2 = \pm 1$	-1
$\omega_{\sigma^*}$	$(\sigma^*)^2 = \pm 1$	-1
$\omega_I$	$I^2 = \pm 1$	-1

TABLE I. Quantum numbers labeling different projective representations of the symmetry group  $p6m^*$ .

Since the semions obey the fusion rule  $s \times s = 1$ , the possible projective representations of  $p6m^*$  it can carry is classified by  $H^2(p6m^*, \mathbb{Z}_2) = H^2(p6m, \mathbb{Z}_2) = \mathbb{Z}_2^4$ , and are therefore labeled by four  $\mathbb{Z}_2$  variables<sup>56</sup>  $\omega_{12}$ ,  $\omega_{\mu^*}$ ,  $\omega_{\sigma^*}$  and  $\omega_I$ , where  $\omega_{12}$  denotes the commutation relation fractionalization<sup>52</sup>  $T_1 T_2 = \pm T_2 T_1$  and the other three variables  $\omega_X$  denote quantum number fractionalization  $X^2 = \pm 1$ , as listed in Table I.

These  $2^4 = 16$  different ways of symmetry fractionalization listed in Table I are all consistent with the semion fusion rule. However, as we will show in the rest of the section, most of them are anomalous and can only be realized on the 2D surface of a 3D SPT state. In other words, for an SET order to be realizable in 2D models, it needs to satisfy anomaly-free conditions which greatly restricts possible ways to fractionalize crystal symmetries. First, if the semion carries a half-integer spin representation, it must also carry fractionalized crystal symmetry quantum numbers  $\omega_{\mu^*} = \omega_{\sigma^*} = \omega_I = -1$ . Second, if the model has odd number of spin- $\frac{1}{2}$  per unit cell, the semion must carry a half-integer spin and  $\omega_{12} = -1$ . Combining these two arguments we can show that there is only one anomaly-free SET order for a fully symmetric chiral spin liquid state on the kagome lattice.

We begin with an argument showing that in an anomaly-free SET order in which the semion carries half-integer spin, it must also carry fractional quantum numbers  $\omega_{\mu^*} = \omega_{\sigma^*} = \omega_I = -1$ . Here our argument is based on the flux-fusion anomaly test developed by Hermele and Chen<sup>48</sup>, which can also be used to restrict crystal symmetry fractionalization visons can carry in a  $\mathbb{Z}_2$  spin liquid<sup>57</sup>. In this test, we put the chiral spin liquid state on a quasi-1D torus as shown in Figs. 1(c) and 1(f), where

the spin liquid state has two nearly degenerate ground states, each corresponds to one type of anyon flux going through the torus. We denote the two states with a trivial anyon flux and a semion flux as  $|1\rangle$  and  $|s\rangle$ , respectively.

If the semion carries a half-integer spin, it implies that it carries a half-integer charge of the  $U(1)$  subgroup generated by  $S^z$ . Hence the ground state  $|1\rangle$  can be smoothly turned to  $|s\rangle$  by adiabatically threading in a  $2\pi$  flux of this  $U(1)$  subgroup along the longitudinal direction in the Hamiltonian. If the Hamiltonian is a Heisenberg model, threading a  $\phi$  flux is achieved by changing the off diagonal coupling to  $J_{ij}(e^{i\phi\frac{y_i-y_j}{L_y}}S_i^+S_j^- + e^{-i\phi\frac{y_i-y_j}{L_y}}S_i^-S_j^+)$ , where  $y_i$  is the transverse coordinate and  $L_y$  is the circumference of the torus in the transverse direction. For a general Hamiltonian, this is achieved by applying the unitary transformation

$$U(\phi) = \prod_j e^{i\phi\frac{y_j}{L_y}S_j^z}. \quad (5)$$

If  $\phi$  is smoothly varied from 0 to  $2\pi$ , the two ground states are adiabatically connected to each other. This method has been applied in DMRG studies to compute the second ground state from the previously obtained one<sup>14,15,58,59</sup>.

Now we consider a  $\mathbb{Z}_2$  symmetry operation  $X$  satisfying  $X^2 = 1$ . As explained in Sec. II, if the semion  $s$  carries a nontrivial symmetry fractionalization  $X^2 = -1$ , the ground state  $|s\rangle$  belongs to a different SPT state protected by  $X$ , as compared to  $|1\rangle$ . [This requires that  $Xx = +x$  ( $Xx = -x$ ) if  $X$  is antiunitary (unitary), respectively.] The key argument of this anomaly test is that if a symmetry  $X$  is preserved throughout the process of flux insertion, the two ground states must belong to the same 1D SPT phase protected by  $X$ . This in turn implies that the semion carries a trivial symmetry fractionalization of  $X$ .

We begin with the example of  $M^* = MT$ , where  $M$  is any mirror operation (for example  $M$  can be either  $\mu$  or  $\sigma$  on the kagome lattice). Because  $M^*$  is an antiunitary operation, we need to set up the torus such that the mirror axis is along the longitudinal direction. Since  $T$  commutes with  $U(\phi)$  and  $M$  maps  $y$  to  $-y$ ,  $M^*$  maps  $U(\phi)$  in Eq. (5) to  $M^*U(\phi)M^* = U(-\phi)$  and hence reverses the threaded flux. Therefore the symmetry  $M^*$  is not preserved in the process of threading the flux.

However, the combined symmetry operation  $M^*e^{i\pi S^x}$  (here the operation  $e^{i\pi S^x}$  denotes the global symmetry operation  $\prod_j e^{i\pi S_j^x}$ ) is preserved during the flux threading, because  $S^x$  anticommutes with  $S^z$ , and a  $\pi$ -rotation with respect to  $S^x$  again reverses the threaded flux. Consequently from the anomaly test argument we conclude that the semion must carry a trivial symmetry fractionalization  $(M^*e^{i\pi S^x})^2 = +1$ . Since the semion also carries a half-integer spin, it fulfills  $(e^{i\pi S^x})^2 = -1$ . Furthermore, the spin rotation  $e^{i\pi S^x}$  must commute with

$M^*$ <sup>60</sup>. From this, we conclude that the semion must also carry a fractionalized quantum number of  $(M^*)^2 = -1$ .

Next, we consider the inversion symmetry  $I$ . Similarly,  $I$  commutes with  $S^z$  and maps  $y$  to  $-y$ . This means that inversion  $I$  is not preserved during the flux threading. However, the combination  $Ie^{i\pi S^x}$  is preserved. Hence, the semion must carry  $(Ie^{i\pi S^x})^2 = +1$  and consequently  $I^2 = -1$ .

In summary, if the semion carries half-integer spin, it must also carry fractionalized crystal symmetry quantum numbers  $(M^*)^2 = -1$  and  $I^2 = -1$ . For example, on the kagome lattice, the semion must carry  $\omega_{\mu^*} = \omega_{\sigma^*} = \omega_I = -1$ .

Furthermore, if the lattice model has odd number of spin- $\frac{1}{2}$  per unit cell, it can be shown that<sup>61,62</sup> the semion must carry a half-integer spin and  $\omega_{12} = -1$ . First, semion, as the only type of nontrivial anyon, must carry a half-integer spin to screen the background spin- $\frac{1}{2}$  quantum number per unit cell. Second, the screening requires a background anyon charge of odd number of semions per unit cell. This means that if a semion moves around an unit cell, it sees a  $\pi$  flux due to the self-statistics of the semion. Hence the semion carries  $\omega_{12} = -1$ . Therefore in lattice models like spin- $\frac{1}{2}$  models on the kagome lattice (with full spin rotation symmetry) the crystal symmetry fractionalization carried by the semion is completely fixed as summarized in the last column of Table I. Consequently, there is one possible SET phase of a symmetric chiral spin liquid.

#### IV. DETECTING CRYSTAL SYMMETRY FRACTIONALIZATION IN DMRG RESULTS

In this section we provide numerical confirmation for the theoretical arguments given in Sec. III. Furthermore, the method presented here can be applied to systems realizing different topological orders or when the assumptions made in Sec. III are not met (for example, in models that lack the  $SU(2)$  spin-rotational invariance, or on lattices with an even number of sites per unit cell).

For this purpose we study the following  $SU(2)$  invariant Hamiltonian on the kagome lattice

$$H = \cos \theta \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sin \theta \sum_{i,j,k \in \Delta, \nabla} \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k), \quad (6)$$

where  $\mathbf{S}_i$  is a spin- $\frac{1}{2}$  operator acting on site  $i$  of the lattice,  $\langle \dots \rangle$  denotes nearest neighbors and sites  $i, j, k$  in the second sum are ordered clockwise around every elementary triangle of the kagome lattice. The Hamiltonian (for non-zero  $\theta$ ) breaks time-reversal  $T$  and mirror symmetries  $\sigma$ ,  $\mu$  but remains invariant under combinations  $\sigma^* = \sigma T$ ,  $\mu^* = \mu T$ . It is also invariant under the reflection  $I$  with respect to the center of the hexagon (see Figs. 2 and 3). Hence, the Hamiltonian has the  $p6m^*$  symmetry discussed in Sec. III.

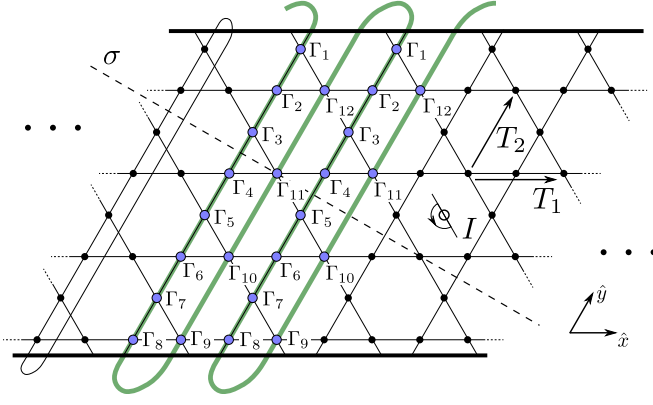


FIG. 2. (Color online) XC8-4 infinite cylinder used to extract  $\omega_{12}$ ,  $\omega_{\sigma^*}$  and  $\omega_I$  quantum numbers. Semi-transparent green line represents an MPS path that covers the infinite cylinder. An MPS unit cell consists of  $N = 12$  tensors  $\Gamma_j$  ordered along the path, see Eq. (9) for comparison. Matrices  $\Lambda_j$  are omitted from the figure to improve clarity. An MPS unit cell (or its multiple) is compatible with translation  $T_2$  and mirror reflection  $\sigma$ . Inversion  $I$  maps an MPS unit cell to itself, provided it is “shifted” by two position along the path. All the symmetry operations act as permutations of lattice sites within (possibly enlarged) MPS unit cell.

Ref. 13 provides compelling evidence that the Hamiltonian in Eq. (6), considered with  $0.05\pi \leq \theta \leq 0.5\pi$ , realizes chiral  $\mathbb{Z}_2$  semion topological order. In our present study we take one particular value of the parameter  $\theta = 0.2\pi$  for concreteness. The identification of an intrinsic topological order was performed by analyzing two ground states on an infinite cylinder found by DMRG<sup>13</sup>. Naturally, these ground states found by DMRG are the minimum entropy states<sup>29</sup>  $|\mathbf{1}\rangle$  and  $|s\rangle$ , which has an anyon flux of  $\mathbf{1}$  and  $s$  threading through the cylinder, respectively. The anyon flux threading through the cylinder in each ground state can be determined by studying the  $SU(2)$  symmetry properties of the corresponding entanglement spectrum<sup>13,30</sup>. Namely, the ground state  $|\mathbf{1}\rangle$  ( $|s\rangle$ ) gives rise to the entanglement spectrum that transforms under integer (half-integer) representations of  $SU(2)$ . This characterization identifies  $|s\rangle$  from the two ground states, which will be used here to further measure the symmetry fractionalization.

Our goal here is to extract all four quantum numbers  $\omega_{12}$ ,  $\omega_{\sigma^*}$ ,  $\omega_{\mu^*}$  and  $\omega_I$  from the ground state wave function  $|s\rangle$  given by DMRG. It is done by computing their corresponding 1D SPT invariants<sup>53,63</sup>, as detailed below. In order to compute  $\omega_{12}$ ,  $\omega_{\sigma^*}$ ,  $\omega_I$  we find  $|s\rangle$  on an XC8-4 infinite cylinder (see Fig. 2), while for the calculation of  $\omega_{\mu^*}$  we resort to the ground state on XC8 infinite cylinder shown on Fig. 3. Here we use the nomenclature introduced in Ref. 64.

Let us consider the ground state  $|s\rangle$  as a strictly 1D translationally invariant state on an infinite chain. Its

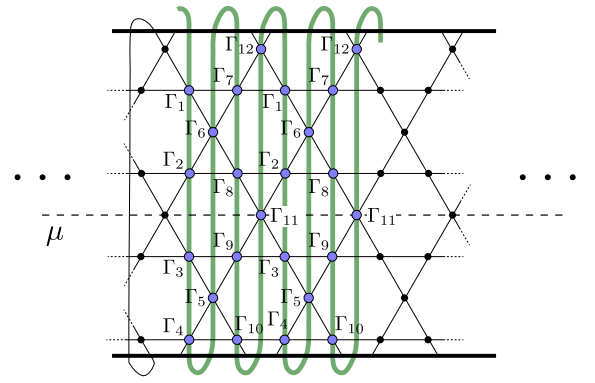


FIG. 3. (Color online) The computation of  $\omega_{\mu^*}$  quantum number form an XC8 infinite cylinder. Mirror reflection  $\mu$  maps an MPS unit cell to itself. An infinite cylinder is covered by an MPS path illustrated with a semi-transparent green line. Matrices  $\Lambda_j$  are not shown for simplicity.

matrix-product representation reads

$$|s\rangle = \cdots \begin{array}{c} \Gamma \\ \downarrow \end{array} \begin{array}{c} \Lambda \\ \downarrow \end{array} \begin{array}{c} \Gamma \\ \downarrow \end{array} \begin{array}{c} \Lambda \\ \downarrow \end{array} \begin{array}{c} \Gamma \\ \downarrow \end{array} \begin{array}{c} \Lambda \\ \downarrow \end{array} \cdots, \quad (7)$$

where rank-3 tensor  $\Gamma$  and matrix  $\Lambda$  are chosen to fulfill canonical relations

$$\begin{array}{c} \Gamma \\ \downarrow \end{array} \begin{array}{c} \Lambda \\ \downarrow \end{array} = \left[ \begin{array}{c} \Gamma \\ \downarrow \end{array} \right], \quad \begin{array}{c} \Lambda \\ \downarrow \end{array} \begin{array}{c} \Gamma \\ \downarrow \end{array} = \left[ \begin{array}{c} \Lambda \\ \downarrow \end{array} \right]. \quad (8)$$

Here,  $\Lambda$  is a diagonal matrix with entries  $\Lambda_{\alpha,\alpha} = \lambda_{\alpha}$ , where  $\lambda_{\alpha}$  are the Schmidt coefficients introduced in Eq. (3). Tensor  $\Gamma$  represents a matrix-product state (MPS) unit cell that is a result of covering an infinite cylinder

$$\begin{array}{c} \Gamma \\ \downarrow \end{array} = \begin{array}{c} \Gamma_1 \\ \downarrow \end{array} \begin{array}{c} \Lambda_1 \\ \downarrow \end{array} \begin{array}{c} \Gamma_2 \\ \downarrow \end{array} \begin{array}{c} \Lambda_2 \\ \downarrow \end{array} \cdots \begin{array}{c} \Lambda_{N-1} \\ \downarrow \end{array} \begin{array}{c} \Gamma_N \\ \downarrow \end{array}, \quad (9)$$

where  $N$  is the size of an MPS unit cell.

It follows from Eq. (8) that the identity  $\mathbf{1}$  is the right eigenvector of a transfer matrix

$$\mathcal{T} \equiv \begin{array}{c} \Gamma \\ \downarrow \end{array} \begin{array}{c} \Lambda \\ \downarrow \end{array} \begin{array}{c} \Gamma^* \\ \downarrow \end{array} \begin{array}{c} \Lambda \\ \downarrow \end{array} \quad (10)$$

to the eigenvalue  $g = 1$ . We have numerically verified that  $\mathbf{1}$  is the only eigenvector of  $\mathcal{T}$  with eigenvalue  $|g| = 1$ .

The  $\omega_{12}$  quantum number is determined by the momentum per unit length of  $|s\rangle$  in Eq. (7). It is thus given by the dominant eigenvalue  $g_{T_2}$  of the generalized transfer matrix

$$\mathcal{T}_{T_2} \equiv P_{T_2} \begin{array}{c} \Gamma \quad \Lambda \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \Gamma^* \quad \Lambda \end{array} = \quad (11)$$

where  $P_{T_2}$  is a permutation of lattice sites within the MPS unit cell that corresponds to a translation in  $\hat{y}$  direction by one lattice unit cell (see Fig. 2). We stress that the translational invariance in  $\hat{y}$  direction is not built in the MPS ansatz for a ground state on an infinite cylinder. It is only (approximately) recovered as the energy optimization of tensors in Eq. (7) reaches convergence. We have observed numerically that the dominant eigenvalue  $g_{T_2} = -1$  for a converged ground state. From this we conclude that  $w_{12} = -1$ , in agreement with prediction of Sec. III.

We now turn to the computation of  $\omega_{\sigma^*}$ . A mirror symmetry  $\sigma$  is chosen such that it does not exchange left and right sides of an infinite cylinder and is compatible with the MPS covering of the cylinder, see Fig. 2. Note that  $\sigma$  is an internal symmetry of the ground state  $|s\rangle$  in Eq. (7) if the MPS unit cell is enlarged to include  $N = 48$  lattice sites. Since the ground state  $|s\rangle$  is invariant under the combination  $\sigma^* = \sigma T$ , the tensor  $\Gamma$  in Eq. (7) transforms in the following way<sup>55</sup>

$$\begin{array}{c} \Gamma^* \\ \text{---} \\ t \\ \text{---} \\ P_\sigma \end{array} = e^{i\theta_{\sigma^*}} \cdot \begin{array}{c} U_{\sigma^*} \Gamma U_{\sigma^*}^\dagger \end{array}, \quad (12)$$

where  $t \equiv \prod_j e^{i\pi S_j^y}$  together with conjugating the tensor  $\Gamma$  implements the time-reversal operation.  $P_\sigma$  is a permutation of lattice sites within (enlarged) MPS unit cell that corresponds to a mirror reflection  $\sigma$ .  $U_{\sigma^*}$  is a unitary matrix that commutes with  $\Lambda$  in Eq. (7) and  $\theta_{\sigma^*}$  is a non-universal phase. It follows from Eq. (12) that the matrix  $U_{\sigma^*}$  is the leading right eigenvector of the generalized transfer matrix

$$\mathcal{T}_{\sigma^*} \equiv \begin{array}{c} \Gamma^* \quad \Lambda \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \Gamma \quad \Lambda \end{array} \quad (13)$$

By iterating Eq. (12) twice, it can be shown that  $U_{\sigma^*}^\dagger U_{\sigma^*}^T$  is the right eigenvector of the transfer matrix  $\mathcal{T}$  in Eq. (10) to the eigenvalue  $g = 1$ . As noted before, identity  $\mathbf{1}$  is the only eigenvector of  $\mathcal{T}$  to the eigenvalue

$g = 1$  which means that  $U_{\sigma^*}^\dagger U_{\sigma^*}^T = \alpha_{\sigma^*} \mathbf{1}$ . Consequently,  $U_{\sigma^*} = \alpha_{\sigma^*} U_{\sigma^*}^T$ , where  $\alpha_{\sigma^*} = \pm 1$  is a 1D SPT invariant that corresponds to the  $\omega_{\sigma^*}$  quantum number. Note that Eq. (12) defines  $U_{\sigma^*}$  only up to a phase. It has no impact on the above reasoning, since any phase ambiguity is canceled in the expression  $U_{\sigma^*}^\dagger U_{\sigma^*}^T$ , as emphasized in Sec. II.

We have computed  $U_{\sigma^*}$  as the dominant right eigenvector of the transfer matrix  $\mathcal{T}_{\sigma^*}$  in Eq. (13) and verified that  $U_{\sigma^*} = -U_{\sigma^*}^T$  which shows that  $\omega_{\sigma^*} = -1$ , as expected.

The computation of  $\omega_{\mu^*}$  is performed in a similar manner. First, the ground state  $|s\rangle$  on an infinite XC8 cylinder carrying semion flux is found. We choose mirror symmetry  $\mu$  such that it does not exchange left and right sides of the cylinder, see Fig. 3. The MPS unit cell consists of  $N = 12$  tensors and it is organized in a way that  $\mu$  is an internal symmetry of the ground state  $|s\rangle$  in Eq. (7). The quantum number  $\omega_{\mu^*}$  is deduced by studying the symmetry properties of the right leading eigenvector  $U_{\mu^*}$  of the generalized transfer matrix

$$\mathcal{T}_{\mu^*} \equiv \begin{array}{c} \Gamma^* \quad \Lambda \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \Gamma \quad \Lambda \end{array}, \quad (14)$$

where  $P_\mu$  is a permutation of sites that implements the mirror reflection  $\mu$ . We find that  $U_{\mu^*}$  is antisymmetric, which shows that  $w_{\mu^*} = -1$ .

In order to conclude the detection of symmetry fractionalization of a chiral spin liquid under investigation, we extract  $\omega_I$  quantum number from the ground state  $|s\rangle$  on an XC8-4 infinite cylinder. The inversion  $I$  maps an MPS unit cell to itself, provided that  $\Gamma_j$  and  $\Lambda_j$  in Eq. (9) are “shifted” by two positions along an MPS path, see Fig. 2. For a symmetric ground state  $|s\rangle$ , the tensor  $\Gamma$  in Eq. (7) transforms in the following way

$$\begin{array}{c} \Gamma^T \\ \text{---} \\ P_I \end{array} = e^{i\theta_I} \cdot \begin{array}{c} U_I \Gamma U_I^\dagger \end{array}, \quad (15)$$

where  $P_I$  is a permutation of lattice sites within a “shifted” MPS unit cell that corresponds to the inversion  $I$ .  $U_I$  is a unitary matrix,  $[U_I, \Lambda] = 0$ , and  $\theta_I$  is a phase. In close analogy to the case of mirror symmetry  $\sigma$ , it can be shown that  $U_I$  is the leading right eigenvector of the generalized transfer matrix

$$\mathcal{T}_I \equiv \begin{array}{c} \Gamma^T \quad \Lambda \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \Gamma \quad \Lambda \end{array} \quad (16)$$

and that  $U_I = \alpha_I U_I^T$ , where  $\alpha_I = \pm 1$  is a 1D SPT invariant that corresponds to the  $\omega_I$  quantum number. We have numerically computed  $U_I$  and checked that it is antisymmetric confirming that  $\omega_I = -1$ , as found in Sec. III.

## V. CONCLUSION

In this work, we classified and numerically detected the symmetry fractionalization in chiral spin liquid states in a spin- $\frac{1}{2}$  model on the kagome lattice. Using the recently proposed method of flux-fusion anomaly test<sup>48</sup>, we show that there are only one anomaly-free way to fractionalize crystal symmetries in a chiral spin liquid state, if we assume that there is a full SU(2) spin-rotational symmetry and there is an odd number of spin- $\frac{1}{2}$  per unit cell of the kagome lattice.

Our theoretical argument proves that all symmetric chiral spin liquids found in previous numerical and theoretical studies belong to the same SET phase. In particular, we provide numerical evidence that the ground states obtained by DMRG belong to this SET phase. Moreover, we show in Appendix A that the chiral spin liquid state obtained in the parton construction<sup>18,19</sup> also belongs to the same phase.

In this work, we also provide general definitions for symmetry fractionalization of space-time symmetries for Abelian anyons with fractional statistics. In particular, the definition we provide for unitary crystal symmetries applies consistently to generic Abelian anyons, but differs from the previously adapted definition for the special case of fermionic anyons by a minus sign. Our definition can be used to study symmetry fractionalization in more complicated Abelian topological orders, and we will leave this to future works.

## ACKNOWLEDGMENTS

We thank Xiao-Gang Wen, Jia-Wei Mei and Meng Cheng for enlightening discussions. L.C. acknowledges support by the John Templeton Foundation. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

Note added: Upon completion of this paper, we became aware of an independent work<sup>65</sup>.

### Appendix A: PSG analysis in Abrikosov-fermion construction of chiral spin liquid.

In this appendix we study the symmetry fractionalization in a chiral spin liquid state on the kagome lattice obtained in fermionic parton construction<sup>11,18,19</sup>, using the method of projective symmetry group (PSG) analysis<sup>33</sup>. This approach reproduces the symmetry fractionalization listed in Table I, which is argued to be the only possibility in Sec. III of the main text.

It is well known that the chiral spin liquid, having the same topological order as a  $\nu = 1/2$  FQH state<sup>10</sup>, can be obtained in a parton construction if the fermionic parton

forms a Chern insulator with the Chern number  $n = 2^{11}$ . On the kagome lattice chiral spin liquids have been constructed using this approach in Ref. 18 and 19 and they have been shown to have good variational energies in VMC studies<sup>19</sup>.

Here we choose the parton wave function constructed by Hu *et al.*<sup>19</sup> as an example and study its symmetry fractionalization using PSG analysis. In this construction the spin- $\frac{1}{2}$  degree of freedom on each site is represented by a fermionic parton,

$$\mathbf{S}_i = f_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} f_{i\beta}. \quad (\text{A1})$$

The wave function of a chiral spin liquid is then constructed by Gutzwiller-projecting a mean field parton wave function,

$$|\Psi\rangle = P_G |\Psi_f\rangle, \quad (\text{A2})$$

where  $P_G$  denotes the Gutzwiller projection, which enforces the constraint that there is exactly one parton per site, and  $|\Psi_f\rangle$  is the mean field parton wave function, which is the ground state of the mean field Hamiltonian,

$$H_{\text{MF}} = \sum_{ij} \left( t_{ij} f_{i\alpha}^\dagger f_{j\alpha} + \text{h. c.} \right). \quad (\text{A3})$$

The hopping amplitudes  $t_{ij}$  are given in Fig. 1(a) of Ref. 19 and are shown in Fig. 4(a) here (here we only show the nearest-neighbor hoppings because these are enough to fix the PSG and the next-nearest-neighbor only improves the variational energy without affecting the PSG). Note that because the chiral spin liquid state breaks time-reversal symmetry, the hopping amplitudes  $t_{ij}$  are complex (an complex hopping flux is required to obtain the nontrivial Chern number).

The parton representation given by Eq. (A1) has an SU(2) gauge structure<sup>9,66</sup>, which is broken to U(1) by the mean field Hamiltonian in Eq. (A3). In other words, the invariant gauge group (IGG)<sup>33</sup> of this parton wave function is U(1).

In this construction the IGG does not match the topological order. As a result the projective representation the fermionic parton carries does not contain all information of the projective representation carried by the semion. Particularly the former is described by  $H^2(G, \text{U}(1))$  where the coefficient is given by the IGG = U(1), while the latter is described by  $H^2(G, \mathbb{Z}_2)$ , where the coefficient is  $\mathbb{Z}_2$  because of the fusion rule  $s \times s = 1$ . In comparison, in parton constructions of a  $\mathbb{Z}_2$  spin liquid<sup>33</sup> both the IGG and the fusion rule of the fermionic anyon are  $\mathbb{Z}_2$ , so there is a one-to-one correspondence between the projective representations carried by the fermionic parton and the fermionic anyon.

The quantum number fractionalization  $X^2 = \pm 1$  of an antiunitary symmetry operator  $X$  can still be extracted from the PSG, because  $H^2(\mathbb{Z}_2^X, \text{U}(1)) = H^2(\mathbb{Z}_2^X, \mathbb{Z}_2) = \mathbb{Z}_2$ . Namely this symmetry fractionalization can be extracted from how  $X$  acts on the parton operator  $X^2 f_{i\alpha} =$



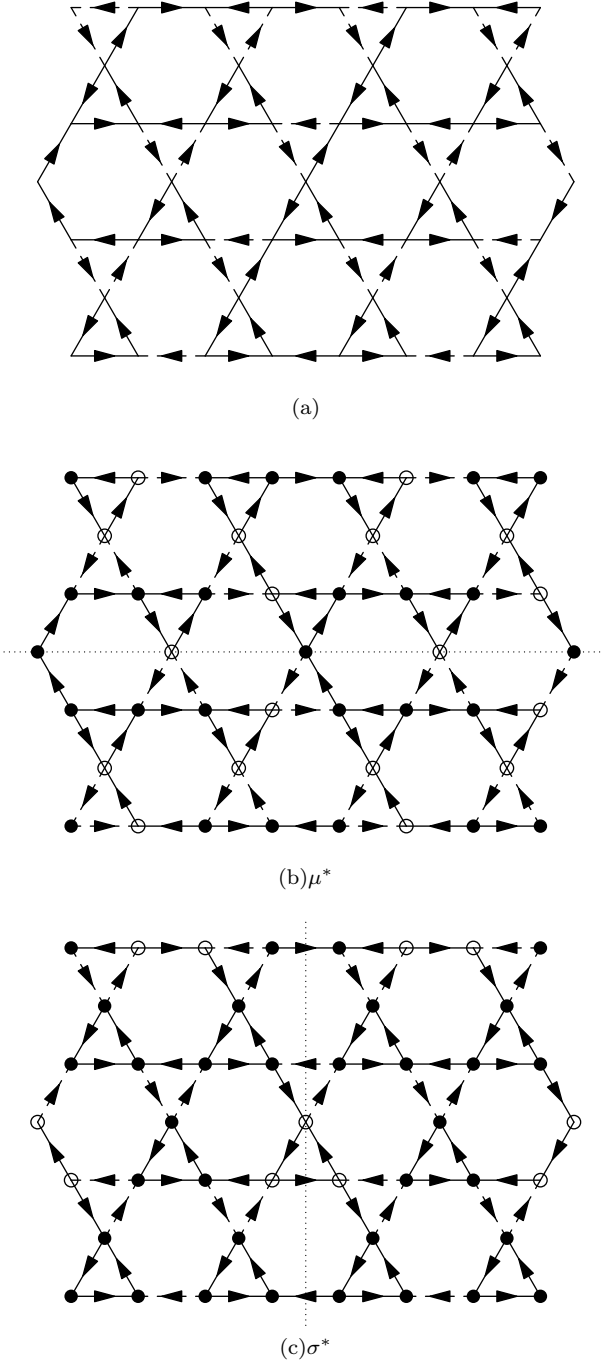


FIG. 4. Mean field ansatz and its PSG. Solid bonds and dashed bonds indicate hopping amplitudes  $t_0$  and  $-t_0$ , respectively. The arrows on the bonds mark the direction along which the hopping is  $\pm t_0$ , while the hopping against the direction of the arrows is  $\pm t_0^*$ . (a) The nearest-neighbor hopping amplitudes in the mean field ansatz in Ref. 19. (b) The hopping amplitudes after the symmetry operation  $\mu^*$ . (c) The hopping amplitudes after the symmetry operation  $\sigma^*$ . In (b) and (c), the mirror axis is marked by a dotted line. The hopping amplitudes can be restored to the ones in (a) by a gauge transformation in Eq. (A4), where  $\eta_i = \pm 1$  on sites marked by filled (open) circles, respectively.

$\omega_X f_{i\alpha}$ , because for an antiunitary operator  $X$  the  $\omega_X$  obtained this way is invariant under a  $U(1)$  gauge transformation and distinguishes different PSGs. For the mean field ansatz presented in Ref. 19, the parton  $f_{i\alpha}$  carries the following projective representation of  $\mu^* = \mu T$ ,

$$\mu^* : f_{i\alpha} \rightarrow \eta_{\mu^*}(i) \epsilon_{\alpha\beta} f_{\mu(i)\beta}, \quad (\text{A4})$$

where  $\mu(i)$  denotes the site that is the mirror image of site  $i$ , and  $\eta_{\mu^*}(i)$  is a site-dependent phase factor. Up to a  $U(1)$  gauge transformation (which does not affect the result of  $\omega_{\mu^*}$ ), the choice of  $\eta_{\mu^*}(i)$  shown in Fig. 4(b) leaves the mean field Hamiltonian in Eq. (A3) invariant. It is easy to check this choice satisfies  $\eta_{\mu^*}(\mu(i))\eta_{\mu^*}^*(i) = +1$ . Therefore from Eq. (A4) we can derive that  $(\mu^*)^2 f_{i\alpha} = -f_{i\alpha}$ , which is consistent with the conclusion of  $\omega_{\mu^*} = -1$  obtained in the main text as listed in Table I.

Similarly the projective representation of  $\sigma^* = \sigma T$  carried by the parton  $f_{i\alpha}$  has a similar form of Eq. (A4), with another site-dependent factor  $\eta_{\sigma^*}(i)$ , as shown in Fig. 4(c). This factor also satisfies  $\eta_{\sigma^*}(\sigma(i))\eta_{\sigma^*}^*(i) = +1$ , and this implies  $\omega_{\sigma^*} = -1$ , which is also consistent with the conclusion in Table I of the main text.

Furthermore, the commutation relation fractionalization  $T_1 T_2 = \pm T_2 T_1$  can also be determined from the parton PSG. It is well known that the parton carries a nontrivial commutation relation fractionalization  $T_1 T_2 = -T_2 T_1$  as the parent gapless  $U(1)$  spin liquid state is known as a  $\pi$ -flux state<sup>67</sup>, indicating that the hopping amplitudes of the fermionic parton has a  $\pi$ -flux in each unit cell (this can be easily checked from Fig. 4). In the main text we argued that this is the only possibility in the chiral spin liquid state.

However, the symmetry fractionalization  $I^2 = \omega_I = \pm 1$  (see the definition in Table I of the main text) cannot be directly derived from the parton PSG, because the parton does not have a nontrivial projective representation of inversion as  $H^2(\mathbb{Z}_2, U(1))$  is trivial. In other words, as the IGG is  $U(1)$ , the parton's projective representation of inversion has a  $U(1)$  phase ambiguity  $e^{i\theta}$  and correspondingly the action of  $I^2$  has a phase ambiguity of  $e^{2i\theta}$ . Thus the two projective representations  $I^2 = \pm 1$  are smoothly connected and cannot be distinguished from each other by studying parton PSG.

This symmetry fractionalization  $\omega_I$ , however, can be obtained from this parton construction by studying the inversion eigenvalue of the ground state wave function. In Sec. II of the main text we showed that  $\omega_I$  can be defined as the inversion parity eigenvalue of the two-semion wave function. Because the model has one spin- $\frac{1}{2}$  on each site, the chiral spin liquid state has a nontrivial background anyon charge of one semion per site<sup>61,62</sup>, and consequently  $\omega_I$  can also be detected from the inversion eigenvalue of the ground state wave function on a system with  $(4n+2)$  sites and open boundaries. In this parton construction, such a wave function is obtained by projecting a mean field wave function as in Eq. (A2), where the mean field wave function, as the ground state of the mean field Hamiltonian, can be obtained by filling all

eigenmodes of the Hamiltonian that lies below the Fermi energy,

$$|\Psi_f\rangle = \prod_{\lambda, E_\lambda < 0} f_{\lambda\uparrow}^\dagger f_{\lambda\downarrow}^\dagger |0\rangle, \quad (\text{A5})$$

where  $\lambda$  labels all eigenmodes of the free mean field Hamiltonian in Eq. (A3) and the product runs over all  $\lambda$  with a negative energy  $E_\lambda < 0$ . Note that the Hamiltonian in Eq. (A3) does not act on the spin quantum number, so all modes are doubly degenerate with  $\alpha = \uparrow$  and  $\downarrow$ .

Next, we consider the action of inversion symmetry on the mean field Hamiltonian. We assume that the fermion takes the following projective representation of  $I$  denoted as  $\tilde{I}$ ,

$$\tilde{I} : f_{i\alpha} \rightarrow \eta_I(i) f_{I(i),\alpha}, \quad (\text{A6})$$

where the phases  $\eta_I(i)$  satisfies

$$\eta_I(i) \eta_I(I(i)) = e^{i\Theta}. \quad (\text{A7})$$

Here  $e^{i\Theta}$  is a site-independent phase factor, and the projective representation satisfies  $\tilde{I}^2 = e^{i\Theta}$ . As we discussed before, one can choose different forms of the projective representation  $\tilde{I}$  satisfying  $\tilde{I}^2 = e^{i\Theta}$  with arbitrary phase  $e^{i\Theta}$ , but this phase ambiguity does not affect the result of symmetry fractionalization  $I^2 = \pm 1$  we derive below.

In the mean field theory the symmetry  $\tilde{I}$  can be viewed as a global symmetry under which the parton  $f_{i\alpha}$  and wave functions all carry linear representations. Particularly  $\tilde{I} H \tilde{I}^{-1} = H$  so we assume the eigenmodes are also eigenstates of  $\tilde{I}$ . Furthermore, because the inversion symmetry also commutes with spin rotations, the two degenerate modes  $f_{\lambda\uparrow}$  and  $f_{\lambda\downarrow}$  have the same inversion eigenvalue  $\tilde{I} f_{\lambda\alpha} = \Lambda_\lambda f_{\lambda\alpha}$ . If  $\tilde{I}^2 = e^\Theta$ ,  $\Lambda_\lambda$  can

only take one of the two values  $\Lambda_\lambda = \pm e^{i\Theta/2}$ . Therefore the inversion eigenvalue of the mean field wave function in Eq. (A5) is determined by multiplying all inversion eigenvalues of occupied modes:

$$\tilde{I} |\Psi_f\rangle = \prod_{\lambda, E_\lambda < 0} (\Lambda_\lambda)^2 |\Psi_f\rangle = (e^{i\Theta})^{2n+1} |\Psi_f\rangle. \quad (\text{A8})$$

In the last step we used the fact that there are in total  $(4n+2)$  occupied modes.

Next we study how  $\tilde{I}$  acts on the Gutzwiller projection  $P_G$ . We saw that the  $\tilde{I}$  eigenvalue of the mean field wave function depends on  $e^{i\Theta}$ , which in turn depends on the gauge choice in Eq. (A6). This  $U(1)$  gauge dependence is canceled by the action of  $\tilde{I}$  on  $P_G$ , as the physical spin wave function obtained after the Gutzwiller projection should be gauge invariant. By definition  $P_G$  maps the Fock states of parton operator to spin states,

$$P_G = \prod_i \sum_\alpha |\alpha\rangle_i \langle 0| f_{i\alpha}. \quad (\text{A9})$$

Therefore applying the transformation in Eq. (A6) we conclude that  $P_G$  transforms as the following under  $\tilde{I}$ ,

$$\tilde{I} P_G \tilde{I}^{-1} = (-e^{i\Theta})^{2n+1} P_G, \quad (\text{A10})$$

where the minus sign comes from exchanging the two fermionic parton operators at  $i$  and  $I(i)$  after the inversion.

Combining the results in Eq. (A8) and (A10), we obtain that the eigenvalue of the physical spin wave function is  $(-1)^{2n+1} = -1$ . Therefore using the definition we describe in Sec. II of the main text we conclude that the semions carry a nontrivial symmetry fractionalization  $I^2 = -1$ .

<sup>1</sup> P. Anderson, Mater. Res. Bull. **8**, 153 (1973).

<sup>2</sup> L. Balents, Nature **464**, 199 (2010).

<sup>3</sup> S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865(R) (1987).

<sup>4</sup> X. G. Wen, Int. J. Mod. Phys. B **04**, 239 (1990).

<sup>5</sup> X.-G. Wen, Int. J. Mod. Phys. B **05**, 1641 (1991).

<sup>6</sup> X. G. Wen, Phys. Rev. B **44**, 2664 (1991).

<sup>7</sup> L. D. Landau, E. M. Lifshitz, and E. M. Pitaevskii, *Statistical Physics* (Butterworth-Heinemann, New York, 1999).

<sup>8</sup> S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865(R) (1987).

<sup>9</sup> P. A. Lee, N. Nagaosa, and X.-G. Wen, Rev. Mod. Phys. **78**, 17 (2006).

<sup>10</sup> V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).

<sup>11</sup> X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).

<sup>12</sup> D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, Phys. Rev. Lett. **99**, 097202 (2007).

<sup>13</sup> B. Bauer, L. Cincio, B. Keller, M. Dolfi, G. Vidal, S. Trebst, and A. Ludwig, Nat. Comm. **5**, 5137 (2014).

<sup>14</sup> S.-S. Gong, W. Zhu, and D. N. Sheng, Sci. Rep. **4**, 6317 (2014).

<sup>15</sup> Y.-C. He, D. N. Sheng, and Y. Chen, Phys. Rev. Lett. **112**, 137202 (2014).

<sup>16</sup> Y.-C. He and Y. Chen, Phys. Rev. Lett. **114**, 037201 (2015).

<sup>17</sup> S.-S. Gong, W. Zhu, L. Balents, and D. N. Sheng, Phys. Rev. B **91**, 075112 (2015).

<sup>18</sup> J.-W. Mei and X.-G. Wen, arXiv:1407.0869 [cond-mat.str-el].

<sup>19</sup> W.-J. Hu, W. Zhu, Y. Zhang, S. Gong, F. Becca, and D. N. Sheng, Phys. Rev. B **91**, 041124(R) (2015).

<sup>20</sup> R. Thomale, E. Kapit, D. F. Schroeter, and M. Greiter, Phys. Rev. B **80**, 104406 (2009).

<sup>21</sup> K. Kumar, K. Sun, and E. Fradkin, arXiv:1507.01278 [cond-mat.str-el].

<sup>22</sup> R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

- <sup>23</sup> L. Messio, C. Lhuillier, and G. Misguich, Phys. Rev. B **87**, 125127 (2013).
- <sup>24</sup> S. Bieri, L. Messio, B. Bernu, and C. Lhuillier, Phys. Rev. B **92**, 060407 (2015).
- <sup>25</sup> X. G. Wen, Phys. Rev. B **40**, 7387(R) (1989).
- <sup>26</sup> X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
- <sup>27</sup> A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
- <sup>28</sup> M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
- <sup>29</sup> Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, Phys. Rev. B **85**, 235151 (2012).
- <sup>30</sup> L. Cincio and G. Vidal, Phys. Rev. Lett. **110**, 067208 (2013).
- <sup>31</sup> J.-W. Mei and X.-G. Wen, Phys. Rev. B **91**, 125123 (2015).
- <sup>32</sup> X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).
- <sup>33</sup> X.-G. Wen, Phys. Rev. B **65**, 165113 (2002).
- <sup>34</sup> H. Yao, L. Fu, and X.-L. Qi, arXiv:1012.4470 [cond-mat.str-el].
- <sup>35</sup> A. M. Essin and M. Hermele, Phys. Rev. B **87**, 104406 (2013).
- <sup>36</sup> A. Mesaros and Y. Ran, Phys. Rev. B **87**, 155115 (2013).
- <sup>37</sup> M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, arXiv:1410.4540 [cond-mat.str-el].
- <sup>38</sup> L. Fidkowski, N. Lindner, and A. Kitaev, Unpublished.
- <sup>39</sup> N. Tarantino, N. Lindner, and L. Fidkowski, arXiv:1506.06754 [cond-mat.str-el].
- <sup>40</sup> J. C. Y. Teo, T. L. Hughes, and E. Fradkin, Ann. Phys. **360**, 349 (2015).
- <sup>41</sup> Y.-M. Lu and A. Vishwanath, arXiv:1302.2634 [cond-mat.str-el].
- <sup>42</sup> L.-Y. Hung and Y. Wan, Phys. Rev. B **87**, 195103 (2013).
- <sup>43</sup> X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
- <sup>44</sup> X. Chen, F. J. Burnell, A. Vishwanath, and L. Fidkowski, Phys. Rev. X **5**, 041013 (2015).
- <sup>45</sup> A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
- <sup>46</sup> C. Wang and T. Senthil, Phys. Rev. B **87**, 235122 (2013).
- <sup>47</sup> Y. Qi and L. Fu, arXiv:1505.06201 [cond-mat.str-el].
- <sup>48</sup> M. Hermele and X. Chen, arXiv:1508.00573 [cond-mat.str-el].
- <sup>49</sup> J.-W. Mei and X.-G. Wen, arXiv:1507.03007 [cond-mat.str-el].
- <sup>50</sup> A. M. Essin and M. Hermele, Phys. Rev. B **90**, 121102(R) (2014).
- <sup>51</sup> L. Wang, A. Essin, M. Hermele, and O. Motrunich, Phys. Rev. B **91**, 121103(R) (2015).
- <sup>52</sup> Y. Qi and L. Fu, Phys. Rev. B **91**, 100401(R) (2015).
- <sup>53</sup> M. Zaletel, Y.-M. Lu, and A. Vishwanath, arXiv:1501.01395 [cond-mat.str-el].
- <sup>54</sup> Y.-M. Lu, G. Y. Cho, and A. Vishwanath, arXiv:1403.0575 [cond-mat.str-el].
- <sup>55</sup> F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).
- <sup>56</sup> W. Zheng, J.-W. Mei, and Y. Qi, arXiv:1505.05351 [cond-mat.str-el].
- <sup>57</sup> Y. Qi, M. Cheng, and C. Fang, arXiv:1509.02927 [cond-mat.str-el].
- <sup>58</sup> Y.-C. He, D. N. Sheng, and Y. Chen, Phys. Rev. B **89**, 075110 (2014).
- <sup>59</sup> W. Zhu, S. S. Gong, and D. N. Sheng, Phys. Rev. B **92**, 014424 (2015), arXiv:1410.4883 [cond-mat.str-el].
- <sup>60</sup> Here the action of  $M^*$  and  $e^{i\pi S^x}$  must commute in the projective representation carried by the semion, because in the spin rotation symmetry group  $SO(3)$ ,  $e^{i\pi S^x}$  can be smoothly deformed to the identity element, which commutes with  $M^*$ .
- <sup>61</sup> M. P. Zaletel and A. Vishwanath, Phys. Rev. Lett. **114**, 077201 (2015).
- <sup>62</sup> M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, arXiv:1511.02263 [cond-mat.str-el].
- <sup>63</sup> F. Pollmann and A. M. Turner, Phys. Rev. B **86**, 125441 (2012).
- <sup>64</sup> S. Yan, D. A. Huse, and S. R. White, Science **332**, 1173 (2011), arXiv:1011.6114.
- <sup>65</sup> M. P. Zaletel, Z. Zhu, Y.-M. Lu, A. Vishwanath, and S. R. White, arXiv:1511.01510 [cond-mat.str-el].
- <sup>66</sup> I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B **38**, 745 (1988).
- <sup>67</sup> M. Hermele, Y. Ran, P. A. Lee, and X.-G. Wen, Phys. Rev. B **77**, 224413 (2008).